Read the following article, from Quanta Magazine[1].

Mathematicians Measure Infinities and Find They're Equal

In a breakthrough that disproves decades of conventional wisdom, two mathematicians have shown that two different variants of infinity are actually the same size. The advance touches on one of the most famous and intractable problems in mathematics : whether there exist infinities between the infinite size of the natural numbers and the larger infinite size of the real numbers.

The problem was first identified over a century ago. At the time, mathematicians knew that « the real numbers are bigger than the natural numbers, but not how much bigger. Is it the next biggest size, or is there a size in between ? » said Maryanthe Malliaris of the University of Chicago, co-author of the new work along with Saharon Shelah of the Hebrew University of Jerusalem and Rutgers University.

In their new work, Malliaris and Shelah resolve a related 70-year-old question about whether one infinity (call it $\mathfrak{p}$) is smaller than another infinity (call it $\mathfrak{t}$). They proved the two are in fact equal, much to the surprise of mathematicians.

« It was certainly my opinion, and the general opinion, that $\mathfrak{p}$ should be less than $\mathfrak{t}$, » Shelah said.

Malliaris and Shelah published their proof last year in the Journal of the American Mathematical Society and were honored this past July with one of the top prizes in the field of set theory. [...]

Many Infinities

The notion of infinity is mind-bending. But the idea that there can be different sizes of infinity ? That's perhaps the most counterintuitive mathematical discovery ever made. It emerges, however, from a matching game even kids could understand.

Suppose you have two groups of objects, or two « sets, » as mathematicians would call them : a set of cars and a set of drivers. If there is exactly one driver for each car, with no empty cars and no drivers left behind, then you know that the number of cars equals the number of drivers (even if you don't know what that number is).

1. www.quantamagazine.org/mathematicians-measure-infinities-find-theyre-equal-20170912/

In the late 19th century, the German mathematician Georg Cantor captured the spirit of this matching strategy in the formal language of mathematics. He proved that two sets have the same size, or « cardinality, » when they can be put into one-to-one correspondence with each other – when there is exactly one driver for every car. Perhaps more surprisingly, he showed that this approach works for infinitely large sets as well.

Consider the natural numbers : 1, 2, 3 and so on. The set of natural numbers is infinite. But what about the set of just the even numbers, or just the prime numbers ? Each of these sets would at first seem to be a smaller subset of the natural numbers. And indeed, over any finite stretch of the number line, there are about half as many even numbers as natural numbers, and still fewer primes. Yet infinite sets behave differently. Cantor showed that there's a one-to-one correspondence between the elements of each of these infinite sets.

1	2	3	4	5	...	(natural numbers)
2	4	6	8	10	...	(evens)
2	3	5	7	11	...	(primes)

Because of this, Cantor concluded that all three sets are the same size. Mathematicians call sets of this size « countable, » because you can assign one counting number to each element in each set.

After he established that the sizes of infinite sets can be compared by putting them into one-to-one correspondence with each other, Cantor made an even bigger leap : He proved that some infinite sets are even larger than the set of natural numbers. Consider the real numbers, which are all the points on the number line. The real numbers are sometimes referred to as the « continuum, » reflecting their continuous nature : There's no space between one real number and the next. Cantor was able to show that the real numbers can't be put into a one-to-one correspondence with the natural numbers : Even after you create an infinite list pairing natural numbers with real numbers, it's always possible to come up with another real number that's not on your list. Because of this, he concluded that the set of real numbers is larger than the set of natural numbers. Thus, a second kind of infinity was born : the uncountably infinite.

What Cantor couldn't figure out was whether there exists an intermediate size of infinity – something between the size of the countable natural numbers and the uncountable real numbers. He guessed not, a conjecture now known as the continuum hypothesis.

In 1900, the German mathematician David Hilbert made a list of 23 of the most important problems in mathematics. He put the continuum hypothesis at the top. « It seemed like an obviously urgent question to answer, » Malliaris said.

Cantor's proof

Not All Infinities Are Equal

The Smallest Infinity

Two different infinite sets have the same size when each element of one set can be paired with an element of the second. For example, every even number can be paired with one of the natural numbers. This feature implies that the two sets are the same size.

All natural numbers	Paired 1-to-1	All even numbers
1	Pair	2
2	Pair	4
3	Pair	6
4	Pair	8
↓	To infinity	↓

Unequal Infinites

Are the real numbers — the set that includes every point on a number line — the same size as the natural numbers? If they were, we could make a countably infinite list of all real numbers:

Natural numbers	Real numbers
1	0.1873 ...
2	0.1874 ...
3	0.1875 ...
4	0.1876 ...
↓ To infinity	↓

Yet this list will always be incomplete. To see why, change the value of the first decimal place in the first number, the second decimal place in the second number, and so on:

1	0.2873 ...
2	0.1974 ...
3	0.1885 ...
4	0.1877 ...
↓ To infinity	↓

Now put all those changed numbers together: 0.2987...

This number doesn't appear on our original infinite list. It differs from the first number on the list in the first decimal place, the second number in the second decimal place, and so on. Therefore there are more real numbers than natural numbers. And so there are at least two sizes of infinity: **countable** and **uncountable**.

Lucy Reading-Ikkanda/Quanta Magazine

Questions

So, a one-to-one correspondence (or bijection) is a function between the elements of two sets, where each element of one set is paired with exactly one element of the other set, and each element of the other set is paired with exactly one element of the first set. There are no unpaired elements.
A bijection from the set X to the set Y has an inverse function from Y to X.

1. If $X = \{0; 1; 2; 3; 4; 5\} = [\![0; 5]\!]$, show that

$$f : \begin{cases} X \to X \\ x \mapsto 5 - x \end{cases}$$

is a one-to-one correspondence.

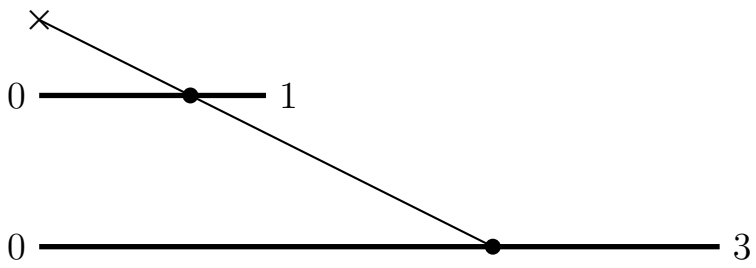
2. Is

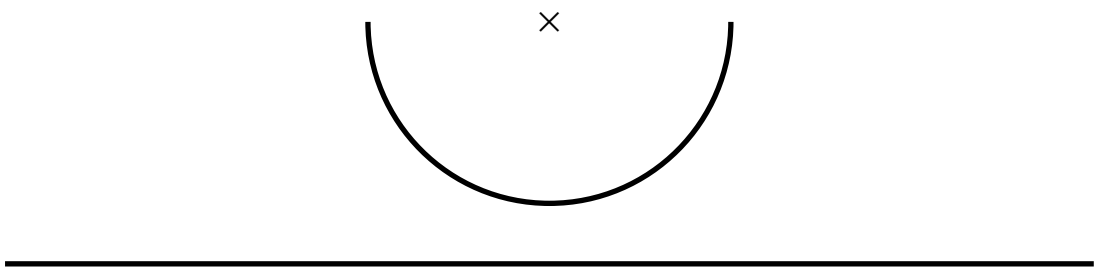
$$g : \begin{cases} \mathbb{R} \to \mathbb{R}^+ \\ x \mapsto x^2 \end{cases}$$

a one-to-one correspondence ?

3. a) Prove that the closed intervals $[0; 1]$ and $[0; 3]$ have the same cardinality.
You might want to find a one-to-one correspondence between these two sets.

b) Here is another way of seeing a one-to-one correspondence :



Why is that ?

4. Use the following drawing to prove that, for any real numbers a and b, the open interval $]a; b[$ has the same cardinality as $\mathbb{R}$: